Modifying Garrison's Math For Four And Five Sided Bamboo Rods

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I began this little exercise as a challenge; I had recently created a single spreadsheet that duplicated Garrison's calculations for six (Hex) sided bamboo rods, and thought it would be easy to modify the math for four (Quad) and five (Penta) sided rods as well. The geometry involved in calculating the volume, weight, and moments of inertia for the Quad and Penta was nothing more than straight forward trigonometry. The resulting stress curves looked good and made it fairly easy, although not automatic, to convert Hex rod tapers to Quad and Penta version.

After distributing the spreadsheet to a few people to try out, Bob Maulucci told me that he had found by experience that the taper dimensions on a Quad needed to be in the neighborhood of 90% to 93% of the Hex taper dimensions to have the same "feel" when casting. When I looked closely at the spreadsheet math, I could see that the Quad was actually thicker, flat to flat, than the Hex it was trying to match. Obviously, there was a problem in the stress calculations.

I obtained a copy of "Roark's Formulas for Stress & Strain", 6th ed., by Warren C. Young, published in 1989 by McGraw-Hill. I discovered that stress is related not only to the cross-sectional area but also to the distance of the outermost fibers from the centerline of the object.

The next thing I did was to try to replicate Garrison's factors using the equations in Young's book.

Garrison's equations for stress are:

$$D = (M/(.120 \times f)^{1/3}$$

where D = flat to flat dimension
M = total moment
f = stress

Rearranging terms,

$$f = M/(.120 \times D^{3)}$$

which is the same as the general form of the stress equation in Young. Next, I confirmed Garrison's calculations.

Bending stress,

$$\sigma = \frac{MC}{I}$$

where C is 1/2 D

M is total moment

$$I = \frac{A(6P_1^2 - a^2)}{24}$$

$$A = \frac{6a^2}{4\tan\alpha}$$

$$\alpha = \frac{360°}{2 \times \text{number of sides}}$$

$$P_1 = a$$

$$P_2 = C = \frac{a}{2\tan\alpha}$$

$$a = 2C\tan\alpha = D\tan\alpha$$

Reducing A,

$$A = \frac{6(2C\tan\alpha)^2}{4\tan\alpha} = \frac{6D^2\tan\alpha}{4} = 0.86603\ D^2$$

$$\sigma = \frac{MC}{I} = \frac{MD}{2I} = \frac{MD}{2\frac{A(5P_1^2)}{24}} = \frac{12MD}{5A\ P_1^2}$$

$$\sigma = \frac{12MD}{5\ (.86603D^2)\ a^2} = \frac{12M}{5(.86003D^3)(1/3)} = \frac{8.3138M}{D^3} = \frac{M}{0.12028\ D^3}$$

Which is very close to Garrison's factor; Since Garrison was using slide rules and hand calculators to do his calculations, it's not surprising he rounded off the factor from 0.12028 to 0.120.

Based on the equations in Young, the stress calculation for a Quad is much simpler than for a Hex. For bending stress

$$\sigma = \frac{MC}{I}$$

where C is again D/2

$$I = \frac{a^4}{12}$$

and $a = D = 2C$

$$\sigma = \frac{Ma}{2\,I} = \frac{12Ma}{2\,a^4} = \frac{6M}{a^3} = \frac{M}{0.16667a^3}$$

The Penta calculations are a bit more complex.

$$I = \frac{A(6P_1^2 - a^2)}{24}$$

where $A = \dfrac{5a^2}{4\,\mathrm{Tan}\,\alpha}$ Since $\alpha = 36°$, $A = 1.72048a^2$

$$P_1 = \frac{a}{2\,\mathrm{Sin}\,\alpha} = \frac{a}{1.17557}$$

$$P_2 = \frac{a}{2\,\mathrm{Tan}\,\alpha} = \frac{a}{1.45309}$$

C is the distance from the center to the farthest point, so $C = P_1$ should be correct; however Young offers another equation for the distance of the farthest fiber from the center when n (number of sides) is odd:

$y_1 = p_1 \cos\left[\alpha \dfrac{(n+1)}{2} - \pi/2\right]$ which yields $y_1 = C = 0.809017\ a$

and define D as the flat to apex distance, which is $D = P_1 + P_2$

Substituting yields

$$I = \frac{1.72048}{24}\,a^2\left[6\left(\frac{a}{2\mathrm{Sin}\,\alpha}\right)^2 - a^2\right] = 0.23955\ a^4$$

Since $D = P_1 + P_2 = 1.53884\ a$,

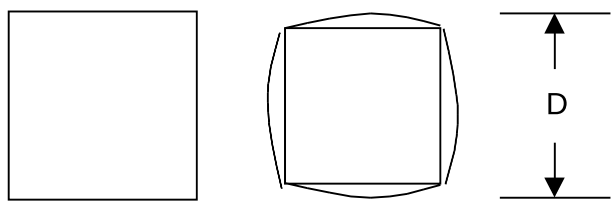
$$\frac{I}{C} = \frac{0.23955\ a^4}{0.809017\ a} = 0.296101\ a^3$$

Substituting D for a will then yield

$$\frac{I}{C} = \frac{(0.296101)D^3}{(1.53884)^3} = 0.081256\ D^3$$

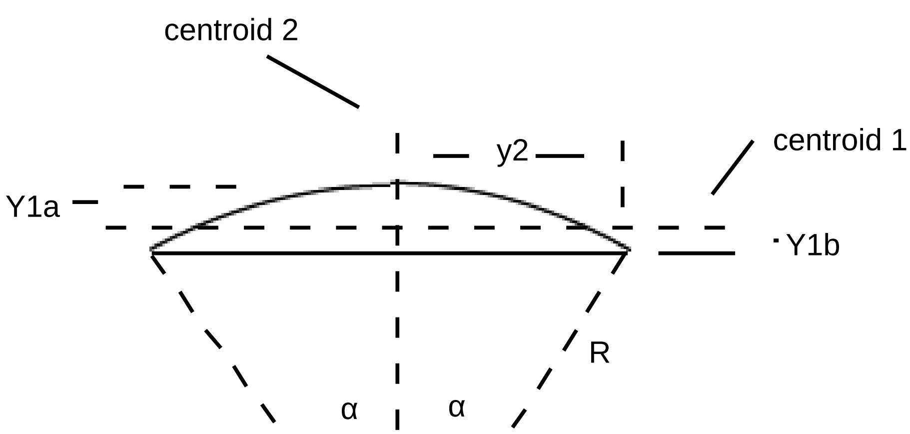
$$\sigma = \frac{M}{0.081256D^3}$$

Some correspondence with Bob Nunley revealed that he usually made his Quads with the natural rounding of the bamboo culm left in place. In other words, instead of a perfectly square cross-section, his Quads were more of a "rounded square". If the above formula for a Quad was used to convert a specific Hex taper, it would not be exactly correct using the rounded square method. This is because the rounded square has a bit less fibers at the outer limit than does the perfect square, so the fibers that were there would see more stress than expected.

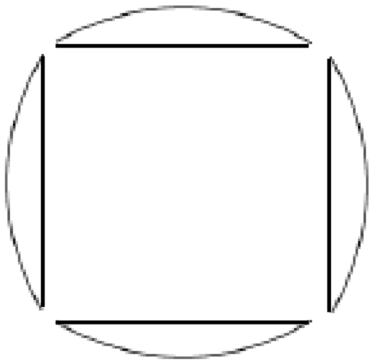


Perfect square Rounded Square

What I needed to do was to find or figure out the formulas to make a rounded square equivalent to a perfect square. Obviously, the flat to flat dimension, D, would need to be bigger on the rounded square. Young provides formulas for a shape called Segment of Solid Circle.



This is the shape of each rounded area that is added to a square in a rounded square (height of rounded area increased for visual emphasis).



The areas on each side use centroid 2 and y_2, while the top and bottom segments use centroid 1 and y_{1a} and y_{1b}. Young's equations are

$$A = \frac{2R^2\alpha^2(1-.2\alpha^2 + 0.019\alpha^4)}{3}$$

where R is the culm radius
α is half the angle subtended by the segment

$$y_{1a} = 0.3R\alpha^2(1 – 0.0976\alpha^2 + 0.0028\alpha^4)$$

$$y_{1b} = 0.2R\alpha^2(1 – 0.0619\alpha^2 + 0.0027\alpha^4)$$

$$y_2 = R\alpha(1 – 0.1667\alpha^2 + 0.0083\alpha^4)$$

$$\text{Inner square area} = (2Y_2)^2$$

$$I_1 = 0.01143R^4\alpha^7(1-0.3491\alpha^2 + 0.0450\alpha^4)$$

$$I_2 = 0.1333R^4\alpha^5(1-0.4762\alpha^2 + 0.1111\alpha^4)$$

$$I_{\text{inner square}} = \frac{a^4}{12} = \underline{(2Y_2)^4}$$

σ

$$C = Y_2 + Y_{1a} + Y_{1b}$$

For a starting point, I have assumed that the average culm is a 2 inch diameter (R = 1), although it does make a difference which will be explained later. Selecting a station with a strip width of 0.215, the circumference (ΠD) is dividing by this width to determine the exact number of strips that could be made from a 2 inch culm (29.2 in this case). This is divided into 360° to determine α.

$$\alpha = \frac{360}{(2 \times 29.2)} = 6.159° = 0.1075 \text{ radians}$$

Using the Quad equations above,

$I = 0.000178063 \quad C = 0.1075$

From the Stress Spreadsheet, the total Moment applied on a Hex rod at this station is 160.6404999, so the stress at this station is

$$\sigma = \frac{MC}{I} = \frac{160.64 \times 0.1075}{0.000178} = 96981$$

The actual stress from the spreadsheet for the Hex rod is 97735. The Quad stress could be made closer in value to the Hex by going to more significant digits for the station dimension, but a dimension of 0.214963 is not practical to achieve, so the 96981 is considered “close enough.”

Substituting into the formulas above yields

$Y_{1a} = 0.003462978$
$Y_{1b} = 0.002309605$
$Y_1 = 0.005772583$
$Y_2 = 0.107294101$
$2*Y_2 = 0.214588201$
$I_1 = 1.88865E-09$
$I_2 = 1.90319E-06$

Calculating the combined I for the two side pieces and the square is easy: it’s merely the sum of the individual Moments for the two sides and the square. I was not able to determine an exact value for the combination of the top and bottom segments plus the square. So far in this analysis, I have not had to “cheat” and make any assumptions; all formulas have been exact and correct. The way I have handled this is to determine the centroid for the top segment (bottom is equivalent), and add that value to the inner square, thereby converting the combined area to a rectangle. The centroid of the top segment is a distance Y_{1b} above the square. The rectangle therefore has sides ($2Y_2$) and ($2Y_2 + 2Y_{1b}$).

Young’s formula for I for a rectangle is

$$I = \frac{bh^3}{12}$$

where b = rectangle base, and h = rectangle height.

Substituting values yields

$$I_{rect} = \frac{2Y_2(2Y_2 + 2Y_{1b})}{12} = 0.000188361$$

and I_{total} is therefore $I_{rect} + 2I_2 = 0.000192167$

C (the distance from the center to the farthest fiber) is also changed

$$C_{rounded} = Y_2 + Y_{1a} + Y_{1b} = 0.113066683$$

The stress is therefore

$$\sigma = \frac{MC}{I} = \frac{160.64 \times 0.11307}{0.0001884} = 94517$$

Which is less stress than the perfect square Quad or the Hex, implying a stiffer action. Looking at the individual segment calculations, I_2 has significantly more impact on the total than does I_1, which implies that my assumption of rectangle is not very close; in other words, intuitively, the top and bottom rounded areas should have much more influence on the stress than the two side rounded areas.

As a check on my assumption, I can use an approximation formula. Young mentions on p. 61, "A closely approximate formula…for the section modulus I/C of any solid section of compact form (e.g., approximately square, circular, triangular, or trapezoidal) is

$$\frac{I}{C} = \frac{A^2}{6.15b}$$

Where A is the area and

b is the maximum width of the section

Using this formula, stress is

$$\sigma = \frac{MC}{I} = \frac{160.64 \times 6.15 \times 0.22613}{0.04935324^2} = 91720$$

which is a bit lower stress than either the perfect square Quad or the Hex. An error in <u>both</u> of the rounded Quad stress calculations above is my assumption that the Total Moment is the same for the rounded square Quad as the perfect square quad. As an example, when the stress curves for the Hex and the perfect square Quad are matched, the flat to flat dimensions are 0.244 and 0.215, respectively; the total moments applied are 170.77 and 160.64, respectively. Cross-sectional areas for the Hex and perfect square Quad are 0.5156 and 0.4935, respectively, which is an 11.5% increase in area, and the increase in total moment is 6.3%. Since the rounded Quad has 6.8% more cross-sectional area, it can be assumed that the total moment will also be higher than that of the perfect square. An increase in the rounded Quad moment of only 3% would change the stress from to 94517 to 97353.

When the flat to flat dimension of the rounded Quad is compared to the flat to flat dimension for the perfect square Quad, the numbers are 0.2261 and 0.215, respectively. To make sure my calculations were responding at smaller taper dimension, I checked all calculations using station 20; Hex flat to flat is 0.123, perfect square Quad is 0.110, and rounded Quad is 0.113, providing stresses of 124602, 124869, and 123037, respectively.

My conclusion is that a rounded Quad can be made to have the same stress curve as a perfect square Quad by making the rounded Quad taper 5% greater at station 60, 2.5% greater at station 20, and this percentage tapering linearly over the entire rod. The rod stress spreadsheet has a section added where this is accomplished automatically; by adjusting a perfect square Quad taper to match stress curves with a Hex rod taper, the spreadsheet will calculate a rounded Quad taper as well.

Some recent discussions on the Rodmakers list about hollow-built rods made me curious about the stress implications. Young has the following equations for hollow beams:

Hollow Quads:

$$A = a_o^2 - a_i^2$$

where a_o is the length of an outer side
a_i is the length of an inner side

$$I = \frac{(a_o^4 - a_i^4)}{12}$$

Since C = D/2, and a = 2C, stress for a hollow Quad becomes

$$\sigma = \frac{MC}{I} = \frac{Ma}{2I} = \frac{12Ma}{2(a_o^4 - a_i^4)} = \frac{6MD}{(a_o^4 - a_i^4)}$$

In the spreadsheet model, Garrison's factor X_2 is the "average" area of the rod cross section, therefore

$$\sigma = \frac{6MD}{(X^2 - a_i^2)}$$

and it's easy to see that if the inner area is set to zero (no hollow), the equation matches the previous equations for a perfect Quad.

For hollow Penta and Hex shapes, Young's formulas are

$$A = nat(1- (t \tan\alpha)/a)$$

$$p_1 = a/(2 \sin \alpha)$$

$$p_2 = a/(2 \tan \alpha)$$

where a is a flat width
α is 360°/(2 n)
n is number of sides
t is wall thickness

If n is odd, $y_1 = y_2 = C = p_1 \cos (\alpha ((n+1)/2) - \pi/2)$

if n/2 is odd, $y_1 = p_1$, $y_2 = p_2$

if n/2 is even, $y_1 = p_2$, $y_2 = p_1$

$$I = na^3t/8(1/3 + \frac{1}{\tan^2 \alpha})[1 - \frac{3 t \tan\alpha}{a} + 4((t \tan\alpha)/a)^2 - 2((t \tan\alpha)/a)^3]$$

Making appropriate substitutions, the Hollow Penta stress equations condense to

$$\frac{I}{C} = 0.72678 D^2 t [1 - \frac{3.3541 t}{D} + \frac{5 t^2}{D^2} - \frac{2.79509 t^3}{D^3}]$$

and stress is therefore

$$\sigma = \frac{MC}{I} = \frac{M}{0.72678 D^2 t [1 - \frac{3.3541 t}{D} + \frac{5 t^2}{D^2} - \frac{2.79509 t^3}{D^3}]}$$

As a check, I set $t = p_2$, which means the thickness is the same as the strip, and therefore there is no hollow. The equation factors down to the same values as the solid Penta equations earlier.

For the hollow Hex equations, the appropriate substitutions yields

$$\frac{I}{C} = 0.96225\, D^2\, t \left[1 - \frac{3\,t}{D} + \frac{4\,t^2}{D^2} - \frac{2\,t^3}{D^3}\right]$$

and stress is therefore

$$\sigma = \frac{MC}{I} = \frac{M}{0.96225\, D^2\, t \left[1 - \frac{3\,t}{D} + \frac{4\,t^2}{D^2} - \frac{2\,t^3}{D^3}\right]}$$

This equation also matches the solid Hex equation when I set $t = D/2$ to make the walls thick enough there is no hollow.